

A density version for Häggström's theorem

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Abstract

Given invariant percolation on a regular tree, where the probability of an edge to be open equals p , is it always possible to find an infinite self-avoiding path along which the density of open edges is bigger than p ?

Let S be an invariant percolation on the edges of the d -regular tree, where the probability of an edge being open equals p . We think of S as an invariant process with values in $\{0, 1\}$ (1 corresponds to open edges). For $\bar{x} = (x_0, x_1, x_2, \dots)$ an infinite self-avoiding path, let $D(\bar{x})$ be the density of the percolation along \bar{x} , that is,

$$D(\bar{x}) = \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n S(x_{k-1}, x_k)$$

and let

$$D(S) = \sup_{\bar{x}} D(\bar{x}) .$$

In general, this is a random variable that is F_∞ -measurable, where F_∞ is the tail σ -algebra. We may look at the essential supremum of this random variable and define

$$D_d(p) = \inf_S \text{ess sup } D(S) ,$$

where the infimum is taken over all invariant percolation distributions on the d -regular tree. For background on invariant percolation, see e.g. [1].

Obviously, $D_d(p)$ is monotone in p and $D_d(p) \geq p$.

Question 1. *Is $D_d(p) > p$ for any $d \geq 3$ and $0 < p < 1$?*

More generally we may ask

Question 2. *What is $D_d(p)$?*

In his seminal paper [2], Olle Häggström proved that any invariant percolation on the d -regular tree, with marginal at least $\frac{2}{d}$, has an infinite cluster. In particular, we get that $D_d(\frac{2}{d}) = 1$ and specifically $D_3(\frac{2}{3}) = 1$.

Theorem 3. $D_3\left(1 - \frac{1}{\sqrt{3}}\right) \geq \frac{1}{2}$.

Proof. Take two iid samples from the percolation distribution and look at their maximum. If $p \geq 1 - \frac{1}{\sqrt{3}}$ then this new percolation has marginal $\geq \frac{2}{3}$, so by Häggström's theorem there is an infinite cluster a.s. and in particular there is \bar{x} with all the edges open. At least one of the two original percolations must have $D(\bar{x}) \geq \frac{1}{2}$, so $D(S) \geq \frac{1}{2}$ \square

More generally, define

$$a(d, k) = 1 - \sqrt[k]{1 - \frac{2}{d}}.$$

Theorem 4. $D_d(a(d, k)) \geq \frac{1}{k}$

Proof. The same proof as the previous theorem, except that you take k copies and work on the d -regular tree. \square

Notice that for $d = 3$ and $2 \leq k \leq 5$ we have $a(d, k) < \frac{1}{k}$, so we get that $D_3(p) > p$ for any $p \in [a(3, k), \frac{1}{k})$, but if $k \geq 6$ then $a(3, k) > \frac{1}{k}$, so we obtain no new information.

However, for $d \geq 4$ we have $a(d, k) < \frac{1}{k}$ for all k , so we get some that $D_d(p) > p$ for any $p \in \cup_{k=1}^{\infty} [a(d, k), \frac{1}{k})$.

In fact,

Theorem 5. For any $d \geq 4$ and any $0 < p < 1$ we have $D_d(p) > p$.

Proof. All we need to do is show that for $d \geq 4$ we have $\cup_{k=1}^{\infty} [a(d, k), \frac{1}{k}) = (0, 1)$. We claim that for any $d \geq 4$ and any $k \geq 1$ we have $a(d, k) \leq \frac{1}{k+1}$ which means that these intervals are overlapping.

Now

$$1 - \sqrt[k]{1 - \frac{2}{d}} \leq \frac{1}{k+1}$$

is equivalent to

$$\left(1 - \frac{1}{k+1}\right)^k \leq 1 - \frac{2}{d}$$

and the left hand side is decreasing (as a function of k) so the maximum is obtained for $k = 1$ and it is $\frac{1}{2} \leq 1 - \frac{2}{d}$. \square

Theorem 6. For any d , the function D_d is uniformly continuous.

Proof. Fix d . Let B be bernoulli percolation on the d -regular tree with marginal ε . For a path of length n the probability of getting at least a 1's is bounded by

$$\binom{n}{an} \varepsilon^{an} \leq (2\varepsilon^a)^n.$$

Since there are $d(d-1)^{n-1}$ paths of length n we get that when $a > \frac{\log(2(d-1))}{\log(1/\varepsilon)}$ the probability of a path with an 1's decays exponentially. We conclude that

$$D(B) \leq f_d(\varepsilon) := \frac{\log(2(d-1))}{\log(1/\varepsilon)}.$$

We now claim that if $0 \leq p < q \leq 1$ and $q - p \leq \varepsilon$ then $D_d(q) - D_d(p) \leq f_d(3\varepsilon)$. This implies uniform continuity since $f_d(\varepsilon) \rightarrow 0$ when $\varepsilon \rightarrow 0$.

To show the claim, let S be an invariant percolation with marginal p and B bernoulli percolation with marginal 3ε . Let S' be their maximum. Then S' has marginal $p + (1-p)3\varepsilon \geq q$ (since we may assume that $p \leq 2/3$, for $p > 2/3$ we have $D_d(p) = D_d(q) = 1$). Therefore, with positive probability, there is an infinite path \bar{x} such that the density of s' along \bar{x} is at least $D_d(q)$. But the contribution of B to the density of \bar{x} is at most $f_d(3\varepsilon)$, so the density of S along \bar{x} is at least $D_d(q) - f_d(3\varepsilon)$. \square

In particular, $D_d(p) \rightarrow 1$ as $p \rightarrow \frac{2}{3}$ so for some $a < \frac{2}{3}$ we have $D_3(p) > p$ for all $p \in [a, 1)$. However, we still don't know that $D_3(p) > p$ for all $0 < p < 1$ and specifically that $D_3(1/2) > (1/2)$.

When $d \rightarrow \infty$ we have that if $p = \frac{1}{d} + \frac{1}{d^2}$ we have $1 - (1-p)^2 = 2p - p^2 > \frac{2}{d}$ so again we have $D(p) \geq \frac{1}{2}$. This works for any fixed k , so if we define the limit

$$D_\infty(x) = \lim_{d \rightarrow \infty} D\left(\frac{2}{d}x\right)$$

we know that $D_\infty(x) \geq \frac{1}{k}$ for any $x > \frac{1}{k}$.

Question 7. *Is it true that $D_\infty(x) = x$?*

Note that the same methods apply to site percolation on regular trees. However, in that case, as $d \rightarrow \infty$ the threshold in Häggström's theorem tends to $\frac{1}{2}$ rather than 0. Indeed, the tree is a bipartite graph and the partition into two sides is invariant, hence we can define a percolation that choose one of the sides with equal probabilities and then put 1s on this side and 0s on the other. This gives a marginal of $\frac{1}{2}$ and also density of $\frac{1}{2}$ along any self-avoiding path. This percolation is ergodic, but have a nontrivial tail σ -algebra.

Question 8. *What can be said about site percolation on regular trees if we require that the tail σ -algebra is trivial?*

We may also consider more general processes, i.e. not $\{0, 1\}$ -valued.

Question 9. *Is it true that for any invariant, non-constant process S on the edges of a regular tree, $D(S) > \mathbb{E}[S(e)]$, where e is some/any edge of the tree?*

An interesting side question is this:

Question 10. *Is it true that when you replace the \limsup in the definition of $D(\bar{x})$ by \liminf you get the same function? If not, do our result still hold for the \liminf version?*

Remark: Häggström's theorem was extended to nonamenable Cayley graphs [1], all the discussion above adapts to this set up.

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References

- [1] I. Benjamini, R. Lyons, Y. Peres and O. Schramm, Group-invariant percolation on graphs. *Geom. Funct. Anal.* 9 (1999), no. 1, 29-66.
- [2] O. Häggström, Infinite clusters in dependent automorphism invariant percolation on trees. *Ann. Probab.* 25 (1997), no. 3, 1423-1436.